JOURNAL OF APPROXIMATION THEORY **18**, 1–8 (1976)

Weakly Proximinal Sets

MICHAEL EDELSTEIN

Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada

Communicated by Philip J. Davis

Received September 13, 1974

1. INTRODUCTION

1.1. A set S in a metric space X is called proximinal [9, p. 704] if every $x \in X$ has a nearest point in S. By analogy we could say that S is densely proximinal if the set of x which has a nearest point in S is dense in X. When X is a Banach space and each closed subset is densely proximinal then X is said to have the property of admitting nearest points [10]. The property of dense proximinality can be weakened in a natural manner when X is a normed linear space by replacing density in the norm topology with density in the weak topology. Similarly, when X is a conjugate space one could consider density in the weak*-topology. The terms weakly proximinal and weak*-proximinal (instead of the awkward weakly-densely-proximinal, etc.) will thus be understood to mean that points in X with a nearest point in S form a weakly, resp. a weak-star, dense set in X.

1.2. If X is reflexive then, as is well known, any weakly closed set is proximinal; hence every closed convex set is proximinal. Similarly, if X is a conjugate Banach space then any weak*-closed set is proximinal. On the other hand any nonreflexive Banach space contains a closed subspace which fails to be proximinal. For, as shown by James [7], in such spaces there is always a continuous linear functional which fails to attain its supremum on the closed unit ball; $f^{-1}[o]$ is readily seen to have the property that no point in its complement has a nearest point in it. Sets failing to be proximinal in this extreme manner are referred to as very-non-proximinal [9]. Examples of bounded convex bodies which are very-non-proximinal are also known (cf. [5]). However, boundedness does eliminate very-non-proximinality from a wide class of nonreflexive Banach spaces. For example, if X is a separable conjugate Banach space and $S \subset X$ is closed and bounded (not necessarily convex) then, as shown in [6], for every $d > 0$ there is an $x \in X$ which has a nearest point s in S and $\| x - s \| = d$.

1.3. In what follows we shall consider the class of Banach spaces X having the following property:

(σ) Every closed and bounded convex subset of X is the closed convex hull of its strongly exposed points.

Recall that $x \in S$ is said to be a strongly exposed point of S if an $f \in X^*$ exists with $f(x) > f(s)$ for every $s \in S$, $s \neq x$, and whenever $\{x_n\} \subset S$ and $f(x_n) \to f(x)$ then $x_n \to x$. One of the main results of this paper is (essentially) that every closed and bounded convex subset of a Banach space having property (σ) is weakly proximinal.

2. On Property (σ)

In [1] Asplund proved the following

2.1. Proposition. *If E is an* SDS *and K is a weak*-compact convex subset of E^* then K is the weak*-closed convex hull of all those of its points which are strongly exposed by functionals from E.*

An SDS, as defined by Asplund [1], is a Banach space X with the property that every convex function defined on X is Fréchet differentiable on a dense G_δ subset of its domain of continuity; (here convex functions are considered which are defined on X with values in $(-\infty, \infty]$, finite valued and continuous on a nonempty set called the domain of continuity).

It should be noted that in the definition of property (σ) closed-and-boundedness rather than weak*-compactness is used; however the set of strongly exposed points there is larger, in general, than the corresponding one in Proposition 2.1. This notwithstanding we show below that Proposition 2.1 implies that any space which is the conjugate of an SDS has property (σ). (An independent and entirely different proof of this fact is given in [4].)

2.2. Proposition. *If $X = E^*$ where E is an* SDS *then X has property* (σ).

Proof. Let C be a closed and bounded convex set in X and W the closed convex hull of the strongly exposed points of C. We have to show that $C = W$. By Proposition 2.1 the weak*-closed convex hull K of C is the same as that of the set of those points $x \in K$ which are strongly exposed by functionals from E. It readily follows that $W \neq \varnothing$. (Indeed, if the weak*-continuous functional f strongly exposes $x \in K$ then clearly a sequence $\{x_n\} \subset C$ must exist such that $f(x_n) \to f(x)$; hence $x_n \to x$ and therefore, since C is closed, $x \in C$ and x is a strongly exposed point of C). Without loss of generality we may assume that the origin O is in W. Suppose now that, contrary to our assertion, there is a $c \in C$ which is not in W. Then an $f \in X^*$ exists such that

$$\sigma = \sup\{f(x): x \in W\} < 1 = f(c).$$

Let $g \in X^*$ be a weak*-continuous functional such that $g(c) = 1$ and let $T: X \to X$ be an isomorphism of X onto itself with the property that $T(c) = c$ and $T[f^{-1}[o]] = g^{-1}[o]$; (e.g., T can be chosen to be the identity on $f^{-1}[o] \cap g^{-1}[o]$ and such that a given $u \in f^{-1}[o]$ be mapped onto a $v \in g^{-1}[o]$ where both u and v are not in $f^{-1}[o] \cap g^{-1}[o]$). With T so chosen we have $T^*g = f$. Indeed, if $x = \alpha c + y$ where $y \in f^{-1}[o]$ then $T^*g(x) = gT(\alpha c + y) = \alpha g(T(c)) + g(T(y)) = \alpha = f(x)$. Since g is weak*-continuous and $g^{-1}[(\sigma + 1)/2]$ separates the points O, $c \in C$, it follows from Proposition 2.1 that a point $w \in C$ exists such that $T(w)$ is a strongly exposed point of $T[C]$ and $g(T(w)) > (\sigma + 1)/2$. If $h \in X^*$ strongly exposes $T(w)$ then T^*h strongly exposes w. (For if $T(z) \neq T(w)$ is in $T[C]$ then $h(T(W)) > h(T(z))$ and therefore $T^*h(w) > T^*h(z)$; further, if $\{x_n\} \subset C$ is a sequence with $T^*h(w) \to T^*h(w)$ then $h(T(x_n)) \to h(T(w))$ so that $T(x_n) \to T(w)$ and $x_n \to w$.) Hence $w \in W$. On the other hand

$$f(w) = T^*g(w) = g(T(w)) > (\sigma + 1)/2 > \sigma$$

which is incompatible with the fact that $f(x) \leqslant \sigma$ for all $x \in W$. This contradiction shows that $W = C$, proving our assertion.

Remark. In [8], R. R. Phelps has recently characterized property (σ) in terms of several other properties. One such characterization [8, Theorem 9] is that a Banach space X has property (σ) if and only if every bounded subset S of X is dentable. (Recall that $S \subset X$ is said to be dentable if for each $\epsilon > 0$ there is an $s \in S$ such that s is not in the closed convex hull of $\{u \in S: \| u - x \| \geqslant \epsilon\}$.)

3. Two Lemmas

In [6] we used certain properties of the extreme points of the set $M_d = S + d\bar{B}$ where B is the unit ball of a Banach space X, S is a closed and bounded subset of X, and d is an arbitrary positive number. The utility of the above set is due to the fact that if $u \in M_d$ is a boundary point of M_d in [6] u has a nearest point x in S and $\| x - u \| = d = \inf\{\| u - s \|: s \in S\}$. The following lemmas give sufficient conditions for the existence of such u.

3.1. Lemma. *If u is a strongly exposed point of $\overline{M}_d$ then u has a nearest point x (in S) and $\| x - u \| = d$.*

Proof. Suppose $\overline{M}_d$ is strongly exposed at u by the functional f. Then, as can be readily seen (cf. [3]) f strongly exposes both S and $d\bar{B}$. If x and y are the points of S and $d\bar{B}$ which are strongly exposed by f then, clearly, $u = x + y$. Thus $u \in M_d$ and the result follows from [6, Lemma 2].

3.2. LEMMA. *Let C be a closed and bounded convex set in a Banach space X. Let $d > 0$ and $M_d = C + d\bar{B}$. Let F be a flat of finite codimension meeting the interior of M_d and let $M' = F \cap M_d$. Then the set of strongly exposed points of $\overline{M}'$ is contained in M_d.*

Proof. Let z be a strongly exposed point of $\overline{M}'$ and $f \in X^*$ a strongly exposing functional (of z with respect to $\overline{M}'$). Without loss of generality we may assume that $z = 0$ and, therefore,

$$\sup\{f(x) \colon x \in M'\} = 0.$$

It follows from the Hahn–Banach theorem that the restriction of f to F can be extended to a $g \in X^*$ with

$$\sup\{g(x) \colon x \in M_d\} = 0.$$

Let H be a subspace of $g^{-1}[o]$ which is complementary to $g^{-1}[o] \cap F$ and suppose that $\{z_n\}$ is a sequence in M' converging to $z = 0$. For $n = 1, 2,...$ let $x_n \in C$, $y_n \in d\bar{B}$ be such that $x_n + y_n = z_n$. Let $x_n = x_n' + x_n''$, $y_n = y_n' + y_n''$ with $x_n', y_n' \in F$, $x_n'', y_n'' \in H$. To prove the lemma it suffices to show that at least one of the sequences $\{x_n\}$, $\{y_n\}$ has a convergent subsequence. Since $\{x_n''\}$, $\{y_n''\}$ are bounded sequences in the finite-dimensional space H there is clearly no loss of generality in assuming that both converge; and, consequently $\lim_{n\to\infty} x_n'' = -\lim_{n\to\infty} y_n''$. It suffices then to show that $\{x_n'\}$ contains a convergent subsequence. Suppose this is not the case and, therefore, an $\epsilon > 0$ and an increasing sequence of positive integers $\{n_i\}$ exist such that

$$\| x'_{n_i} - x'_{n_j} \| > \epsilon \qquad (i \neq j;\, i, j = 1, 2,...).$$

Since g, like f, strongly exposes O with respect to $\overline{M}'$ it readily follows that a $\delta > 0$ exists such that the set

$$U = \{u \in \overline{M}' \colon g(u) > -\delta\}$$

is of diameter less than $\epsilon/2$.

Let $w \in M_d$ be an interior point of M_d with $g(w) > -\delta/2$ and suppose $r > 0$ is such that $w' \in M_d$ whenever $\| w - w' \| < r$. Let N_1 be such that for $m, n > N_1$ it is always true that

$$\| y_n'' + x_m'' \| = \|(y_n'' - \lim_{k\to\infty} y_k'') + (x_m'' - \lim_{k\to\infty} x_k'')\| < r.$$

The fact that $g(z_n)$ tends to zero as $n \to \infty$ is readily seen to imply that $\{g(x_n')\}$ and $\{g(y_n')\}$ both converge and $\lim_{n\to\infty} g(x_n') = -\lim_{n\to\infty} g(y_n')$. Thus an N_2 exists such that whenever $m, n > N_2$ then

$$| g(y_n' + x_m')| < \delta/2.$$

Let $N = \max(N_1, N_2)$ and suppose $n, n_1, n_2 > N$. Let $w_k' = -(y_n'' + x_{n_k}'')$, $k = 1, 2$, and set

$$u_k = \tfrac{1}{2}(w + w_k') + \tfrac{1}{2}(y_n + x_{n_k}).$$

Then $u_k \in M_d$ since $w + w_k', y_n + x_{n_k} \in M_d$. On the other hand $u_k \in F$ since $u_k = \frac{1}{2}w + \frac{1}{2}(y_n' + x_{n_k}')$; hence $u_1, u_2 \in \overline{M}'$. Now

$$| g(u_k)| = \tfrac{1}{2} | g(w) + g(y_n' + x_{n_k}')| < \delta \text{ so that } u_k \in U.$$

Thus $\frac{1}{2} \| x_{n_1}' - x_{n_2}' \| = \| u_1 - u_2 \| < \epsilon/2$, which is impossible. This contradiction shows that $\{x_n\}$ contains a convergent subsequence completing the proof of the lemma.

4. Main Results

4.1. Theorem. *Let C be a closed and bounded convex set in a Banach space X having property (σ). If $d > 0$ then the set of points x for which a $y \in C$ exists such that*

$$\| x - y \| = d = \inf\{\| x - c \| : c \in C\}$$

is weakly dense in the boundary of

$$M_d = C + d\bar{B}$$

where $B = \{x \in X : \| x \| < 1\}$.

Proof. Let u be an arbitrary point of the boundary of M_d and suppose that u does not belong to M_d. Let W be a weak neighborhood of u (in X). Then a finite set of functionals $\{f_1, f_2, \ldots, f_n\} \subset X^*$ exists such that

$$F = \cap \{f_i^{-1}[1] : i = 1, 2, \ldots, n\}$$

is contained in W and $M' = F \cap M_d$ meets the interior of M_d. By Lemma 3.2 the strongly exposed points of M' belong to M_d. Since these form a nonempty subset of the boundary of M_d it follows that such points may serve as x in the statement of the theorem.

4.4. Corollary. *With X and C as in the preceding theorem, C is weakly proximinal.*

4.3. Proposition. *Let S be a closed and bounded set in a Banach space X having property (σ). Then the set of points in X which have a nearest point in S contains infinitely many closed rays emanating from points of S.*

Proof. Suppose that $f \in X^*$, $\|f\| = 1$, has the property that $f(u) = \sup\{f(x): \|x\| \leqslant 1\}$ for some u, with $\|u\| = 1$ and $f(s) = \sup\{f(x): x \in S\}$ for some $s \in S$. Let R be the ray emanating from s and containing $s + u$. If $v = s + \lambda u$ with $\lambda \geqslant 0$ then $\lambda = \|v - s\| = \inf\{\|v - x\|: x \in S\}$. Indeed, if for some $y \in S \|y - v\| < \lambda$ then $|f(y - v)| < \lambda$ and $f(y) - f(s) = f(y - v) + f(\lambda u) = f(y - v) + \lambda > 0$; this, however, is impossible since $f(y) - f(s) \leqslant 0$ for all $y \in S$. It follows that all points of R have s as a nearest point. Now, if z is a strongly exposed point of $M = \overline{\text{co}}\, S + \bar{B}$ then there is an $f \in X^*$ satisfying the condition stated above and, moreover, if z', z'', are distinct strongly exposed points of M, then clearly, the corresponding rays $R_{z'}$, $R_{z''}$, which are determined by them are also distinct. Thus if the set of strongly exposed points of M is infinite the proposition follows. If not, then $\overline{\text{co}}\, S + \bar{B}$ is the convex hull of a finite set so that S must be compact and the result is obvious.

5. Examples and Remarks

5.1. Every nonreflexive space (whether it has property (σ) or not) contains a closed and bounded convex set which fails to be densely proximinal. For, if $f \in X^*$ is any continuous linear functional which does not attain its supremum on the closed unit ball then $C = \{x \in X: \|x\| \leqslant 1$ and $f(x) = 0\}$ has the desired property. Indeed, if x is any point with $\|x\| < 1$ and $f(x) \neq 0$ then

$$\inf\{\|x - y\|: y \in C\} = \inf\{\|x - y\|: y \in f^{-1}[o]\}$$

but, as mentioned earlier, $f^{-1}[o]$ is very-non-proximinal.

5.2. An open problem mentioned by several authors is whether every locally uniformly convex reflexive Banach space admits nearest points. While we are unable to give a conclusive answer to the above, we do give a negative answer to a closely related question. In the following example we show that a reflexive, strictly convex, separable Banach space exists which fails to admit nearest points.

EXAMPLE 1. Let $X = \ell_2 \oplus \mathbb{R}$ be equipped with the norm $\|\cdot\|$ defined by $\|(x, r)\| = \max(\|x\|_{\ell_2}, |r|)$ and let $|||\cdot|||$ be defined by setting

$$|||(x, r)||| = \|(x, r)\| + \left(r^2 + \sum_{n=1}^{\infty} (x_n^2/2^{2n})\right)^{1/2}$$

where x_n is the nth coordinate of $x = (x_1, x_2, ..., x_n, ...)$. The above norms are equivalent since

$$\|(x, r)\| \leqslant |||(x, r)||| \leqslant 3\,\|(x, r)\|.$$

On the other hand since the functionals sending (x, r) to x_n and r distinguish between members of X, $|||\cdot|||$ is, by known results on renormings, strictly convex; it is obviously reflexive and separable.

Now let $S = \{(e_k, 2 + (1/k)): k = 1, 2, ...\}$ where e_k is the kth member of the standard orthonormal basis for ℓ_2, and let

$$U = \{(u, r): \|u\|_{l_2} < \tfrac{1}{2}, |r| < \tfrac{1}{2}\}.$$

For any (u, r) in the neighborhood U of the origin we then have

$$|||(u, r) - (e_k, 2 + (1/k))|||$$
$$= 2 - r + (1/k) + \left((2 - r + (1/k))^2 + \sum_{n \neq k} (x_n^2/2^{2n}) + ((1 - x_k)^2/2^{2k})\right)^{1/2}$$
$$> 2 - r + \left((2 - r)^2 + \sum_{n=1}^{\infty} (x_n^2/2^{2n})\right)^{1/2} = \inf\{|||(u, r) - w|||: w \in S\}.$$

Thus no point in U has a nearest point in the closed and bounded set S.

To show that X fails to be uniformly convex set $r_k = 2^{-k-1}(2^{2k} - 1)^{1/2}$, $k = 1, 2, ...,$ and $u_k = (\tfrac{1}{2}e_k, r_k)$. With $u_0 = (0, \tfrac{1}{2}) \in X$, it readily follows that $|||u_k||| = 1$, $k = 0, 1, 2, ...,$ $|||u_0 + u_k||| \to 2$ and clearly $u_k \not\to u_0$.

5.3. It would seem natural to ask whether the results of this paper carry over in some fashion to farthest points. The following example serves to show that l_1 which, as a separable conjugate space, has (σ) contains a symmetric closed and bounded convex body C with the property that no point in that space has a farthest point in C.

EXAMPLE 2. Let

$$C = \left\{x = (x_1, x_2, ..., x_n, ...) \in l_1: \|x\| + \left(\sum_{n=1}^{\infty} (x_n^2/2^{2n})\right)^{1/2} \leqslant 1\right\}$$

and suppose $x \in C$. Clearly $\|x\| < 1$, so that $\|y - x\| < \|y\| + 1$ for all

$y \in \ell_1$. Thus to prove the assertion it suffices to show that a sequence $\{x^{(m)}\}$, of points in C, exists such that for any $y \in \ell_1$ $\| y - x^{(m)} \| \to \| y \| + 1$. The following is readily seen to be such a sequence. With $x_k^{(m)}$ denoting the kth coordinate of $x^{(m)}$, set $x_k^{(m)} = 0$ if $k \neq m$ and $x_m^{(m)} = 1 - 2^{-m}$.

5.4. If X is a Banach space containing a symmetric closed and bounded convex body C such that no point of X has a farthest point in C then using C as a new unit ball for a renorming of X the complement of the original open ball is readily seen to be very-non-proximinal. This observation together with the preceding remark shows that a renorming of ℓ_1 exists containing a very-non-proximinal set whose complement is a closed and bounded symmetric convex body.

Acknowledgment

This research was supported by the National Research Council of Canada Grant A-3999.

References

1. E. Asplund, Fréchet differentiability of convex functions, *Acta Mathematica* **121** (1968), 31–49.
2. E. Asplund, Farthest points in reflexive locally uniformly rotund Banach spaces, *Israel J. Math.* **4** (1966), 213–216.
3. J. Collier and M. Edelstein, Common supports for pairs of sets, *Israel J. Math.* **19** (1974), 237–240.
4. J. Collier and M. Edelstein, On strongly exposed points and Fréchet differentiability, *Israel J. Math.* **17** (1974), 66–68.
5. M. Edelstein and A. C. Thompson, Some results on nearest points and support properties of convex sets in c_0, *Pacific J. Math.* **40** (1972), 553–560.
6. M. Edelstein, A note on nearest points, *Quarterly J. Math.* **21** (1970), 403–405.
7. R. C. James, Reflexivity and the supremum of linear functionals, *Ann. of Math.* **66** (1957), 159–169.
8. R. R. Phelps, Dentability and extreme points in Banach spaces, *J. Functional Analysis* **16** (1974), 78–90.
9. I. Singer, Best approximation in normed linear spaces, Centro Internationale Mathematico Estivo, Roma, 1973.
10. D. E. Wulbert, Differential theory of non-linear approximation, preprint.